

THE DUAL SPACE OF AN OPERATOR ALGEBRA

BY

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Introduction. The purpose of this paper is to study noncommutative C^* -algebras as Banach spaces. The Gelfand representation of an abelian C^* -algebra as the algebra of all continuous complex-valued functions on its spectrum has made it possible to apply the techniques of measure theory and the topological properties of compact Hausdorff spaces to the study of such algebras. No such structure theory of general C^* -algebras is available at present. Many theorems about the Banach space structure of abelian C^* -algebras are stated in terms of topological or measure-theoretic properties of their spectra; although much work has been done of late in studying an analogous dual object for general C^* -algebras, the generalization is far from exact. For this reason we shall confine our study primarily to W^* -algebras in which the lattice of self-adjoint projections will be used as a substitute for the Borel sets of the spectrum of an abelian C^* -algebra. Using a theorem of Takeda [15] we shall be able to extend some of our results to general C^* -algebras.

In [10] Sakai proved that any C^* -algebra which is the dual of some Banach space has a representation as a W^* -algebra on some Hilbert space. Dixmier [3] has proved the converse assertion, so it is possible to consider W^* -algebras in a quite abstract fashion. It is this point of view which will predominate in this paper.

Let F be a Banach space and suppose that the Banach space dual F^* of F is a W^* -algebra, which will be denoted by M . In §I we list some theorems and definitions about the topological properties of F and M as well as some related results.

In §II we give a number of characterizations of the weakly relatively compact (abbreviated “wrc”) subsets of F . These are applied to prove a conjecture of Sakai [13] that the Mackey topology of M agrees with the strong* topology on the unit sphere of M (see §I for definitions). We conclude the section with an example which clearly shows the difference between the abelian and nonabelian cases and serves as a counterexample to other possible conjectures.

In §III we move to the dual M^* of M where the situation becomes much more difficult. We are able to extend several of the characterizations of §II to give conditions for weak relative compactness in M^* . Also we give two formulations of the Vitali-Hahn-Saks Theorem for the noncommutative case and mention an open problem which remains in this area.

Certain special algebras are the objects of study in §IV. We obtain a noncommutative version of Phillips’ Lemma [9] for W^* -algebras with sufficiently many

finite-dimensional projections. We also consider dual rings (in the sense of [7]) which are C^* -algebras and show that certain of the properties of Banach spaces considered by A. Grothendieck in [6] hold for such algebras.

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I. Preliminaries. Let M be W^* -algebra and F the unique [11, p. 1.22] Banach space such that $F^* = M$. S denotes the unit sphere of M and P the set of self-adjoint projections in M . We assume the reader is familiar with the general properties of W^* -algebras. F has a natural embedding in M^* and we shall identify F with its image in M^* when convenient.

A positive functional f in M^* is called *normal* if f lies in F . It is called *singular* if there does not exist a positive normal g in M^* such that $f \geq g$. A set K in F is called *invariant* if for each a in M and f in K the functionals $f(a \cdot)$ and $f(\cdot a)$ lie in K . By [11, p. 1.26], if f is in F , then $f = f'_1 - f'_2 + i(f''_1 - f''_2)$, where the functionals f'_1, f'_2, f''_1, f''_2 are positive and unique with the property $\|f'_1 - f'_2\| = \|f'_1\| + \|f'_2\|$ and $\|f''_1 - f''_2\| = \|f''_1\| + \|f''_2\|$. We define $[f] = f'_1 + f'_2 + f''_1 + f''_2$ for any f in F . If F is in M^* , then f has a unique decomposition $f = f^n + f^s$ (or $f_n + f_s$), where f^n is in F and f^s is a linear combination of singular positive functionals [16].

REMARK I.1. We note for the sequel that it is an immediate corollary of the work of Grothendieck [6] that if M is abelian and K is a bounded set in M^* , then K is weakly relatively compact iff for each sequence (p_n) in P of pairwise orthogonal projections $f(p_n) \rightarrow 0$ uniformly for f in K .

We shall abbreviate the phrase "weakly relatively compact" with "wrc."

The *weak* topology* ($\sigma(M, F)$ -topology) on M is the linear topology generated by the seminorms $|f(\cdot)|$ for all f in F . The *strong topology* (s -topology) is generated by the seminorms $|f(a^*a)|^{1/2} = \|a\|_f$ for all positive f in F . The *strong* topology* (s^* -topology) on M is generated by the seminorms $\|a\|'_f = f(a^*a + aa^*)^{1/2}$ for all positive f in F . The *Mackey topology* on M is the topology of uniform convergence on the wrc sets of F .

REMARK I.2. In [11, p. 1.64] Sakai shows that whenever M is represented as a weakly closed algebra of operators on some Hilbert space, the weak* topology of M agrees with the weak operator topology on the bounded sets of M . It follows from this that the s -topology agrees with the strong operator topology on bounded sets of M , and the s^* -topology agrees with the strong* operator topology on bounded sets of M .

II. Weak compactness in F . In this section we shall prove a number of equivalent conditions for a set K in F to be weakly relatively compact, and then we shall

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give some applications—primarily generalizations of some theorems of Sakai [13, 14].

PROPOSITION II.1. *Let f be in M^* . Then f is in M_s^* iff for each nonzero p in P there exists q in P such that $p \geq q > 0$ and $f(q) = 0$.*

Proof. First suppose that f is in M_s^* . Then by [11, p. 1.26], since M^* is the predual of M^{**} , we have $f = f'_1 - f'_2 + i(f''_1 - f''_2)$, where each of f'_1, f'_2 etc. are positive and singular [12] and [16]. Thus, by [17], if we are given $p \neq 0$ in P , we may find q_0 in P such that $0 < q_0 \leq p$ and $f'_1(q_0) = 0$. Similarly find q_1 in P such that $q_1 \neq 0$ and $q_1 \leq q_0$ and $f'_2(q_1) = 0$, etc. Finally we get $q \neq 0$ in P such that $q \leq p$ and $f(q) = 0$.

Now suppose that f satisfies the other condition. We show that f is in M_s^* . By [16] $f = f_n + f_s$ (uniquely), where f_n is in F and f_s is in M_s^* . We must only show that $f_n = 0$. We may assume that the positive part f_{n1} of f_n is $\neq 0$. Let p be the support projection of f_{n1} . It is well known [3, p. 62] that f_n is faithful on p . Choose by the method of the first part of this proof a projection q in P such that $0 < q \leq p$ and $[f_s](q) = 0$. Then $f = f_n$ on q and hence is faithful on q , which contradicts the hypothesis. Q.E.D.

THEOREM II.2. *A bounded set K in F is weakly relatively compact iff one of the following holds:*

- (1) *The restriction of K to each maximal abelian $*$ subalgebra of M is weakly relatively compact.*
- (2) *If $\{p_n\}$ is an orthogonal sequence in P , then $\lim_{n \rightarrow \infty} f(P_n) = 0$ uniformly for f in K .*

REMARK. That (1) implies that K is wrc was first proved by Takesaki [16]. We include a proof of Takesaki's result for completeness.

Proof. By Remark I.1 (1) and (2) are equivalent, so we only need to show that (1) implies that K is wrc, since the reverse implication is trivial. Consider K as lying in M^* , and let f be in the weak* closure of K in M^* . Since K is bounded, its weak* closure is weak* compact; so if we can show that f lies in F (considering F as lying in M^*) we shall have proved the theorem. But by hypothesis the restriction of f to a maximal abelian $*$ subalgebra M' lies in F restricted to M' . Thus f^s is singular, but f^s is normal on each M' . Suppose there exists p in P such that $f^s(p) \neq 0$. Let $\{q_\theta\}$ be a maximal orthogonal family in P such that each q_θ is $\leq p$ and $f^s(q_\theta) = 0$. Then by Proposition II.1 we have that $\text{lub } \{q_\theta\} = p$. If M' is a maximal abelian $*$ subalgebra containing $\{q_\theta\}$ and p , then since f_s is normal on M' , we have $f^s(p) = f^s(\text{lub } \{q_\theta\}) = 0$. This contradicts $f^s(p) \neq 0$. Thus $f^s(p) = 0$ for all p in P , so $f^s = 0$ by the spectral theorem. Q.E.D.

Our next result goes in the other direction in that we give a condition which is superficially stronger than weak relative compactness. Parts of the proof are adaptations of an argument of Sakai [13] for a special case.

THEOREM II.3. *A bounded set K in F is wrc iff: (3) there exists positive g in F such that given $\alpha > 0$ there exists $\beta > 0$ such that if a is in S and $g(a^*a + aa^*) < \beta$, then $|f(a)| < \alpha$ for each f in K .*

Proof. It is immediate that (3) implies (2), so we need only prove the converse. This will be broken up into three lemmas.

LEMMA II.3a. *Suppose $\{a_n\}$ is a self-adjoint sequence in S which converges to 0 in the s -topology. Then given $\delta > 0$ there exists a sequence $\{p_n\}$ in P such that $s: p_n \rightarrow I$, and $\|a_n p_n\| \leq \delta$ for each $n = 1, 2, \dots$*

Proof. Let X be the characteristic function of the interval $(-\delta, \delta)$. Then for each n we define $p_n = X(a_n)$ by the functional calculus for a self-adjoint operator. Thus if $p'_n = I - p_n$,

$$\delta^{-2}(a_n^2) \geq p'_n \geq 0 \quad \text{for each } n.$$

Since $s: a_n \rightarrow 0$, we have that the left member of the inequality converges strongly to 0. Hence $s: p'_n \rightarrow 0$, so $s: p_n \rightarrow I$. Also it is immediate that $\|a_n p_n\| \leq \delta$ for each n . Q.E.D.

LEMMA II.3b. *Let $\{f_n\}$ be a sequence in F converging weakly to some f_0 in F . Suppose $\{a_n\}$ is a sequence in S such that both the sequence $\{a_n\}$ and the sequence $\{a_n^*\}$ converge strongly to 0. Then*

$$\lim_{n \rightarrow \infty} f_i(a_n) = 0 \quad \text{uniformly for } i = 0, 1, 2, \dots$$

Proof. We first make a number of reductions to simplify the proof. Since the $\{f_n\}$ are bounded, we may assume that they lie in the unit sphere of F . Set $f = \sum_{n=1}^{\infty} [f_n] \cdot 2^{-n}$. Let p be the support projection for f . Then $f_n(a) = f_n(pap)$ for any a in M and each $n = 0, 1, \dots$; so we may restrict attention to the algebra pMp , or instead merely assume (as we shall do) that f is faithful on M . The self-adjoint and skew-adjoint parts of the sequence $\{a_n\}$ both converge strongly to 0, so we may assume that each a_n is self-adjoint.

Define a new norm $\| \cdot \|'$ on M by $\|a\|' = f(a^*a)^{1/2}$, a in M . Define a metric $d(,)$ on S by $d(a, b) = \|a - b\|'$. Then by [3, p. 62], the metric d gives the strong topology on S . Thus S is a complete metric space with the metric d .

Let $\alpha > 0$ be given and define

$$H_i = \{a \text{ in } S : |f_j(a) - f_0(a)| \leq \alpha \quad \text{for all } j \geq i\}.$$

Clearly each H_i is strongly closed since the $\{f_n\}$ are strongly continuous on S . Also S is the union of all the H_i for $i = 1, 2, \dots$, since the $\{f_n\}$ converge weakly to f_0 . Thus we may apply the Baire Category Theorem and get that there exists a_0 in S , $\beta > 0$ a real number, and an integer j_0 such that if a is in S and $d(a, a_0) \leq \beta$, then a is in H_{j_0} . That is, under those conditions, $|f_j(a) - f_0(a)| \leq \alpha$ for all $j \geq j_0$. We now apply Lemma II.3a to get that there exists a sequence $\{p_n\}$ in P such that p_n converges strongly to I and $\|a_n p_n\| \leq \alpha/6$, for each $n = 1, 2, \dots$

Therefore, replacing $f_j - f_0$ by g_j , we can write

$$\begin{aligned} |(f_j - f_0)(a_n)| &\leq |g_j(p_n a_n p_n)| + |g_j[(I - p_n)a_n p_n]| + |g_j[p_n a_n (I - p_n)]| \\ &\quad + |g_j[(I - p_n)a_n (I - p_n)]| \\ &\leq \alpha + |g_j[(I - p_n)a_n (I - p_n)]|. \end{aligned}$$

Now put $b_n = p_n a_0 p_n + (I - p_n)a_n(I - p_n)$. Then b_n is in S and

$$\begin{aligned} d(b_n, a_0) &= \|(I - p_n)a_0 p_n + p_n a_0 (I - p_n) + (I - p_n)a_0 (I - p_n) - (I - p_n)a_n (I - p_n)\|' \\ &\leq 3\|I - p_n\|' + \|(I - p_n)a_0 p_n\|'. \end{aligned}$$

Since $(I - p_n) \rightarrow 0$ strongly, we may conclude by [11, p. 1.15] that there exists n_0 such that $n \geq n_0$ implies $\|I - p_n\|' \leq \beta/4$ and $\|(I - p_n)a_0 p_n\|' < \beta/4$. Then for $n > n_0$ we have that $d(b_n, a_0) < \beta$ so

$$|g_j(b_n)| = |g_j(p_n a_0 p_n) + g_j[(I - p_n)a_n(I - p_n)]| \leq \alpha$$

for $j > j_0$ and $n > n_0$. Similarly,

$$d(p_n a_0 p_n, a_0) \leq 3\|I - p_n\|' < \beta$$

for $n > n_0$ and $p_n a_0 p_n$ is in S , hence

$$|g_j(p_n a_0 p_n)| \leq \alpha \quad \text{for } j > j_0 \text{ and } n > n_0.$$

Therefore we have

$$|g_j[(I - p_n)a_n(I - p_n)]| \leq 2\alpha, \quad \text{for } j > j_0 \text{ and } n > n_0.$$

Thus we finally get $|g_j(a_n)| \leq 3\alpha$ for $j > j_0$ and $n > n_0$. Since $\alpha > 0$ was arbitrary, we are done. Q.E.D.

LEMMA II.3c. *Let K be a wrs set in F . Given $\varepsilon > 0$ there exists $\delta > 0$ and a finite subset K' of K such that if a is in S and $[f](aa^* + a^*a) < \delta$, for all f in K' , then $|f(a)| < \varepsilon$ for all f in K .*

Proof. Suppose the lemma is false for some $\alpha > 0$. Then by induction we can construct sequences $\{f_i\}$ in K and $\{a_j\}$ in S such that $|f_{i+1}(a_i)| \geq \alpha$ and $[f_i](a_j^* a_j + a_j a_j^*) < 2^{-j}$ for all $i \leq j$. By the Eberlein-Smulian Theorem [4, p. 430] there is some subsequence of $\{f_i\}$ which is weakly convergent. For notational simplicity we assume that $\{f_i\}$ converges weakly to f_0 . Using the symbols f and p as in the last lemma we have

$$\begin{aligned} f(a_j^* a_j + a_j a_j^*) &= \sum_{i=1}^{\infty} [f_i](a_j^* a_j + a_j a_j^*) 2^{-i} \leq \sum_{i=1}^j 2^{-i} [f_i](a_j^* a_j + a_j a_j^*) + \sum_{i=j+1}^{\infty} \|f_i\| 2^{-i} \\ &\leq (2^{-j}) + \frac{2^{-j}}{1-1/2} \cdot (\sup \{\|f\| : f \text{ is in } K\}). \end{aligned}$$

Therefore $f(a_j^* a_j + a_j a_j^*) \rightarrow 0$ as $j \rightarrow \infty$. By [3, p. 62] we have that the sequence $\{pa_j p\}$ is strong* convergent to 0. Thus Lemma II.3b gives that $|f_i(pa_j p)| = |f_i(a_j)| \rightarrow 0$ (as $j \rightarrow \infty$) uniformly in i . This contradicts $|f_{i+1}(a_i)| \geq \alpha$ for all $i = 1, 2, \dots$. Q.E.D.

Proof of the theorem. Let $\alpha_n = 1/n$ and choose β_n and $K'_n = \{f_1^n, \dots, f_{m_n}^n\}$ for each $n = 1, 2, \dots$ by Lemma II.3c. Set $g = \sum_{n=1}^{\infty} 2^{-n} (\sum_{i=1}^{m_n} 2^{-i} f_i^n)$. It is clear that g is the desired positive element of F . Q.E.D.

The next condition we shall prove is much like (3) but is formally weaker. It is primarily of interest because it can be used as a condition for weak relative compactness in M^* .

THEOREM II.4. *A bounded subset K of F is wrc iff: (4) Each sequence $\{f_i\}$ in K has a subsequence $\{f_{i_n}\}$ such that given $\alpha > 0$ there exists an integer n_0 and a relatively $\sigma(M, F)$ -open set V in S which is nonvoid and such that if a is in V , then*

$$|(f_{i_n} - f_{i_m})(a)| \leq \alpha, \quad \text{for all } n, m \geq n_0.$$

Proof. Suppose K is wrc. If $\{f_n\}$ is a sequence in K , there is a weakly convergent subsequence $\{f_{n_i}\}$ in $\{f_n\}$ by the Eberlein-Smulian Theorem. Given $\alpha > 0$ we define

$$G_n = \{a \text{ in } S : |(f_{i_j} - f_{i_k})(a)| \leq \alpha \text{ for all } j, k \geq n\}.$$

Clearly each G_n is $\sigma(M, F)$ -closed and S is the union of all the G_n . Since S is $\sigma(M, F)$ -compact and Hausdorff for the $\sigma(M, F)$ -topology, an application of the Baire Category Theorem gives the desired integer n_0 and relatively $\sigma(M, F)$ -open subset of S .

Now suppose that K satisfies (4). By the Eberlein-Smulian Theorem we need only to show that for each sequence $\{f_i\}$ in K there is a subsequence $\{f_{i_n}\}$ of $\{f_i\}$ which is wrc. Let $\{f_i\}$ be a sequence in K and choose a subsequence $\{f_{i_n}\}$ by (4). Let $\{p_k\}$ be a sequence in P which decreases down to 0. By Theorem II.2, we need only show that $f_{i_n}(p_k) \rightarrow 0$ (as $k \rightarrow \infty$) uniformly for $n = 1, 2, \dots$. Let $\alpha > 0$ be given, and choose n_0 and V by (4). Let a_0 be a fixed element of V . Set $b_k = p'_k a_0 p'_k + p_k$. Then for some k_0 , if $k > k_0$ we have that b_k is in V and also $p'_k a_0 p'_k$ is in V . Thus if $n > n_0$ and $k > k_0$, $|(f_{i_n} - f_{i_{n_0}})(p_k)| \leq 2\alpha$. Since $f_{i_{n_0}}(p_k) \rightarrow 0$ as $k \rightarrow \infty$ we have the desired uniformity. Q.E.D.

REMARK. It should be noted that Theorem II.3 is a direct generalization of the results of [1] for the case of weak compactness in L_1 spaces. We shall extend the theorem in the next section to give a criterion for weak compactness in the dual space of an arbitrary C^* -algebra, thus giving a complete generalization of the commutative case.

The next corollary is merely a listing of some other equivalent conditions for relative compactness in F . They are all intermediate between conditions (2) and (3) in the sense that each one immediately implies (2) and is implied by (3). Condition (7) is the converse of a result of Takesaki [16] and (9) is the extension of a result of Umegaki [18] to general W^* -algebras.

COROLLARY II.5. *A bounded subset K of F is wrc iff any one of the following conditions is satisfied.*

(5) *If $\{p_n\}$ is an increasing sequence in P , then $\lim_{n \rightarrow \infty} f(p_n)$ exists uniformly for f in K .*

(6) *If $\{a_n\}$ is an increasing bounded sequence of self-adjoint elements of M , then the limit of $f(a_n)$ exists uniformly for f in K .*

(7) *Same as (5) with arbitrary nets instead of sequences.*

(8) *Same as (6) with arbitrary bounded nets instead of sequences.*

(9) *There exists positive g in F such that given $\alpha > 0$ there is $\beta > 0$ such that if p is in P and $g(p) < \beta$, then $|f(p)| < \alpha$ for each f in K .*

(10) *Same as (9) with p being any positive element of S , instead of being restricted to lie in P .*

It is interesting to compare the following theorem with the examples at the end of this section, for in doing so one can see very clearly the exact way that "non-commutativity" enters the picture.

THEOREM II.6. *Let $\{p_\theta\}$ be an arbitrary family of orthogonal projections in P with $\text{lub } \{p_\theta\} = I$. A bounded subset K of F is wrc iff:*

(11) *Given $\alpha > 0$ there is a finite set $p_1, \dots, p_n$ in $\{p_\theta\}$ such that if $p = \text{lub } \{p_1, \dots, p_n\}$ and $p' = I - p$, then $\|R_{p'} L_p f\| \leq \alpha$ for each f in K .*

Proof. Clearly (11) implies (2) so we need only show that if K is wrc, then K satisfies (11).

Let $\varepsilon > 0$ be given. Choose a positive g in F by Theorem II.3 for the set K . Then there exists $\delta > 0$ such that if a is in S and $g(aa^* + a^*a) < \delta$, then $|f(a)| < \varepsilon/4$ for all f in K . Since g is normal there exists a finite set $p_1, \dots, p_k$ in $\{p_\theta\}$ such that if $p = p_1 + \dots + p_k$, then $g(p') < \delta/2$. Thus for any positive a in $p'Mp'$ with $\|a\| \leq 1$ we have $g(a^*a + aa^*) < \delta$. Hence $|f(a)| < \varepsilon/4$ for any positive a in $p'Mp'$ with $\|a\| \leq 1$. Thus for any b in $p'Mp'$ with $\|b\| \leq 1$, $|f(b)| < \varepsilon$. This is equivalent to the assertion of the theorem. Q.E.D.

The next three results are all applications of conditions (1)–(11) above. Theorem II.7 is a conjecture of Sakai [13], and Theorem II.8 and its corollary are extensions of a proposition of Sakai [14].

THEOREM II.7. *The Mackey topology coincides with the strong* topology on S .*

Proof. By [11, p. 1.16] the Mackey topology is stronger than the strong* topology. Thus we need only show the other inclusion.

Suppose K is a wrc subset of F and $\{a_\theta\}$ is a net in S which converges to 0 in the s^* -topology. Let $\alpha > 0$ be given. Then by Theorem II.3 there is a positive g in F and a $\beta > 0$ such that if a is in S and $g(a^*a + aa^*) < \beta$, then $|f(a)| < \alpha$ for each f in F . Thus if we choose θ_0 such that if $\theta \geq \theta_0$, then $g(a_\theta^*a_\theta + a_\theta a_\theta^*) < \beta$; then we may conclude that if $\theta \geq \theta_0$, $|f(a_\theta)| < \alpha$ for all f in K . Thus $\{a_\theta\}$ converges to 0 in the Mackey topology. Q.E.D.

THEOREM II.8. *Let X be a Banach space such that X^* is weakly sequentially complete (i.e., each weakly Cauchy sequence in X^* converges to an element of X^*). Let T be a continuous linear operator from X into F . Then T is weakly compact (i.e., T maps the unit sphere of X into a wrc set in F).*

Proof. Let B be the unit ball of X . By Theorem II.2 we need only show that $T(B)$ is weakly relatively compact when restricted to each maximal abelian $*$ subalgebra N of M . Let N be such a subalgebra. Then N is the dual space of some Banach space G which is a quotient space of F .

Let π be the quotient map of F onto G . If λ is the composition of T and π , we have that λ maps X into G so λ^* maps N into X^* . But it is shown in [4, p. 494] that any such map as λ^* is weakly compact (in fact that any bounded linear map of an abelian C^* -algebra into a weakly sequentially complete Banach space is weakly compact). Thus by [4, p. 485], λ is also weakly compact. But this means exactly that $T(B)$ restricted to N is wrc; thus $T(B)$ is wrc by Theorem II.2. Hence T is weakly compact. Q.E.D.

COROLLARY II.9. *Let N be a C^* -algebra. Then any bounded linear map T of N into F is weakly compact.*

Proof. By a result of Sakai [12], which we shall prove in the next section, N^* is weakly sequentially complete; so the theorem applies to show that T is weakly compact. Q.E.D.

REMARK. It has been conjectured by Sakai [14] that the result of Grothendieck mentioned in the proof of Theorem II.8 extends to arbitrary C^* -algebras. Theorem II.8 seems to be the best result known for the general case at present.

We conclude this section with an example of the pathological properties of the noncommutative case. This has been constructed by many workers in the field and has been published in Sakai [12].

EXAMPLE II.10. Let H be a separable Hilbert space and $B(H)$ the algebra of bounded operators on H . Let $\{x_n\}$ be an orthonormal basis for H , and define operators $\{a_n\}$ in $B(H)$ by:

$$a_n(x_k) = (x_k, x_n)x_1.$$

Dixmier [3] proves that $B(H)$ is the dual of the space $TC(H)$ of trace class operators on H with the duality given by

$$a(f) = \text{tr}(fa)$$

where a is in $B(H)$, f is in $TC(H)$, tr denotes trace, and (fa) is defined by operator multiplication. Now the operators $\{a_n\}$ may be thought of as lying in either $B(H)$ or $TC(H)$. Set $f_n = a_n^*$, and consider the sequence $\{f_n\}$ in $TC(H)$. Clearly $\{f_n\}$ converges weakly to 0 in $TC(H)$ and $\{a_n\}$ converges to 0 in the strong operator topology

of $B(H)$. (Note, however, that $\{a_n^*\}$ does not converge strongly to 0 in $B(H)$.) However, the sequence $\{a_n\}$ does not converge to 0 uniformly on the set $\{f_n\}$ for we have

$$a_n(f_n) = \text{tr}(a_n^* a_n) = 1$$

for each $n=1, 2, \dots$

III. Properties of M^* . In this section we generalize several results from the last section as well as prove by a new method the result of Sakai [12] that F is weakly sequentially complete. We also give a complete generalization of the Vitali-Hahn-Saks Theorem.

THEOREM III.1. *Let $\{f_k\}$ be a sequence in M^* such that $\{f_k\}$ converges to 0 in the weak* topology. Let $f_k = f_k^n + f_k^s$ be the decomposition into normal and singular parts as in [16]. Then $\{f_k^n\}$ converges to 0 in the weak topology of M^* , and hence $\{f_k^s\}$ converges to 0 in the weak* topology.*

Proof. By the definition of the natural embedding of F in M^* , it is clear that $\{f_k^n\}$ converges weakly to 0 iff it converges weak* to 0. Since $\{f_k\}$ converges weak* to 0, the uniform boundedness theorem gives that the sequence $\{f_k\}$ is bounded in M^* ; and hence by [16] the sequence $\{f_k^n\}$ is also bounded in M^* , and similarly for $\{f_k^s\}$. By the spectral theorem, every operator in M can be uniformly approximated by linear combinations of elements of P ; thus we need only prove that $\{f_k^n(p)\}$ converges to 0 for each p in P .

Define $f = \sum_{k=1}^{\infty} [f_k^s] 2^{-k}$, and let p be any projection in P . Note we use [15] in order to apply [11, p. 1.26] to M^* . Note also that if q is in P and $f(q)=0$, then $f_k^s(q)=0$ for each $k=1, 2, \dots$. Let $\{p_\theta\}_{\theta \in \Omega}$ be a maximal family of orthogonal projections in P satisfying:

$$(\text{'}) \quad f(p_\theta) = 0, \quad (\text{''}) \quad p_\theta \leq p$$

for each θ . Then by [17], $\text{lub } \{p_\theta\} = p$. Define set functions Δ_k on the subsets of Ω by

$$\Delta_k(l) = f_k \left(\sum_{\theta \in l} p_\theta \right),$$

where l is any subset of Ω . Then the sequence $\{\Delta_k\}$ consists of bounded finitely additive set functions on Ω (with uniformly bounded total variation since $\{f_k\}$ is bounded); also $\lim_{k \rightarrow \infty} \Delta_k(l) = 0$ for each subset l of Ω . Thus we can apply [2, p. 32] to get that $\lim_{k \rightarrow \infty} \sum_{\theta \in \Omega} |\Delta_k(\{\theta\})| = 0$. Since $\Delta_k(\{\theta\}) = f_k(p_\theta) = f_k^n(p_\theta)$ by ('), we have that $\sum_{\theta \in \Omega} |f_k^n(p_\theta)| \rightarrow 0$ as $k \rightarrow \infty$. Hence we have

$$f_k^n(p) = f_k^n \left(\sum_{\theta \in \Omega} p_\theta \right) = \sum_{\theta \in \Omega} f_k^n(p_\theta) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This gives the theorem. Q.E.D.

THEOREM III.2. *Let M be a C^* -algebra on a Hilbert space H with the property that each self-adjoint element in M can be uniformly approximated by linear combinations of pairwise commuting self-adjoint projections in M . Let N be the weak closure of M in $B(H)$. If p is a self-adjoint projection in N , then p is the limit in the strong operator topology of a net of projections in P .*

Proof. Let p be a self-adjoint projection in N . Let $\alpha > 0$ be given and let $\eta_1, \dots, \eta_n$ be elements of norm 1 in H . We must show that there exists q in P such that $\|(p-q)\eta_i\| < \alpha$ for each $i=1, 2, \dots, r$. By Kaplansky's Density Theorem [8] there exists a net $\{a_\theta\}$ of self-adjoint elements in S which converges strongly to p . Also by Lemma 1 of [8] we have that $\{a_\theta^2\}$ converges strongly to $p^2=p$. Thus we may assume that each a_θ is positive. By the hypothesis we may assume

$$a_\theta = \sum_{i=1}^{n_\theta} \pi_i^\theta p_i^\theta$$

where the p_i^θ are pairwise orthogonal projections in P for each θ and the π_i^θ are positive scalars less than or equal to 1. If $0 < l < 1$, define

$$a_\theta^l = \sum \{\pi_i^\theta p_i^\theta : \pi_i^\theta \leq l, i = 1, \dots, n_\theta\}.$$

It is clear from a simple calculation that both $a_\theta - (a_\theta)^n$ and $a_\theta^l - (a_\theta^l)^n$ are positive operators for each positive integer n and also that

$$a_\theta - (a_\theta)^n \geq a_\theta^l - (a_\theta^l)^n.$$

For fixed n the left-hand side converges strongly to $p - p^n = 0$. Thus the right-hand side converges strongly to 0. But $\lim_{n \rightarrow \infty} \|(a_\theta^l)^n\| = 0$ uniformly in θ since $\|(a_\theta^l)^n\| \leq l^n$ and $0 < l < 1$. Thus $\{a_\theta^l\}$ converges strongly to 0. Now fix l so that $1-l < \alpha/4$. Choose θ_0 such that if $\theta \geq \theta_0$ then $\|a_\theta^l(\eta_i)\| < \alpha/4$ for each $i=1, \dots, r$. Choose $\theta_1 \geq \theta_0$ so that if $\theta \geq \theta_1$, then $\|(p-a_\theta)(\eta_i)\| < \alpha/4$ for each $i=1, \dots, r$.

Fix $\theta \geq \theta_1$ and we get the following inequality for $q = \sum \{p_j^\theta : \pi_j^\theta > l \text{ and } j=1, \dots, n_\theta\}$ and each $i=1, \dots, r$:

$$\begin{aligned} \|(p-q)\eta_i\| &\leq \|(p-a_\theta)\eta_i\| + \|(a_\theta-q)\eta_i\| \\ &\leq \alpha/4 + \|(q-a_\theta+a_\theta^l)\| + \|a_\theta^l(\eta_i)\| < 3\alpha/4 < \alpha. \end{aligned} \quad \text{Q.E.D.}$$

COROLLARY III.3. *F is weakly sequentially complete.*

Proof. Let $\{f_n\}$ be a sequence in F , and consider $\{f_n\}$ as lying in M^* . If $\{f_n\}$ is weakly Cauchy, it is bounded by the uniform boundedness theorem. Thus $\{f_n\}$ is weak* Cauchy and bounded in M^* , so $\{f_n\}$ converges weak* to some $f = f^n + f^s$ in M^* (since bounded sets of M^* are weak* compact). Thus $\{f - f_k\} = \{(f^n - f_k^n) + f^s\}$ converges weak* to 0. By Theorem III.1 we have $(f^n - f_k^n)$ converges weak* to 0. Thus $f^s = 0$, so f is in F , and $\{f_k\}$ converges weakly to f . Q.E.D.

COROLLARY III.4. *Each projection p in M^{**} is the strong limit of a net in P when M is considered as lying in M^{**} by the canonical embedding.*

Proof. This is immediate from [15], Theorem III.2, and the fact [3, p. 57] that the strong topology on the unit sphere is preserved under $*$ isomorphism. Q.E.D.

THEOREM III.5. *If $\{f_n\}$ is a sequence in M_s^* , then any weak* limit point of $\{f_n\}$ lies in M_s^* .*

Proof. Set $f = \sum_{n=1}^{\infty} ([f_k]/\|f_k\|)2^{-k}$. Then if p is in P and $f(p)=0$, then $f_k(p)=0$ for $k=1, 2, \dots$. Also f is in M_s^* , since M_s^* is uniformly closed. Suppose that some net $\{f_n\}$ in $\{f_n\}$ converges weak* to some $g = g^n + g^s$ in M^* . We must show that $g^n=0$; to do that it suffices by the spectral theorem to show that $g^n(p)=0$ for each p in P .

Let p be in P and choose an orthogonal set $\{p_\theta\}$ in P which is maximal with respect to the following properties:

(') The $\{p_\theta\}$ are orthogonal.

(") Each p_θ is $\leq p$.

(") $f(p_\theta) = g^s(p_\theta) = 0$.

This is clearly possible by Zorn's Lemma, and by [17] we have $\text{lub } \{p_\theta\} = p$. Thus if $\{q_\phi\}$ is the net of finite sums of the elements of $\{p_\theta\}$, then

$$\begin{aligned} g^n(p) &= \lim_\phi g^n(q_\phi) = \lim_\phi [g^n(q_\phi) + g^s(q_\phi)] \\ &= \lim_\phi \{\lim_\pi f_\pi(q_\phi)\} = 0, \end{aligned}$$

since $f(p_\theta)=0$ for all θ implies $f(q_\phi)=0$ for all ϕ implies $f_\pi(q_\phi)=0$ for all π . Q.E.D.

Now that we have some preliminary results out of the way we are ready to formulate and prove the noncommutative version of the Vitali-Hahn-Saks Theorem (cf. [4, p. 158] for the formulation in the commutative case).

THEOREM III.6. *Let $\{f_n\}$ be a sequence in F and suppose that $\lim_{n \rightarrow \infty} f_n(p)$ exists for each p in P . Then:*

(') $\sup \{\|f_n\| : n=1, 2, \dots\} < \infty$.

(") There exists f_0 in F such that $\{f_n\}$ converges weakly to f_0 .

(") If g is any positive element of F such that $g(p)=0$ implies $[f_n](p)=0$ for p in P and each $n=1, 2, \dots$ then given $\alpha > 0$ there is $\beta > 0$ such that if p is in P and $g(p) < \beta$, then $|f_n(p)| < \alpha$ for $n=1, 2, \dots$.

Proof. To prove (') it is clearly enough to prove that $\{f_n(a)\}$ is a bounded set of numbers for each self-adjoint a in M (by the uniform boundedness theorem). But the classical Vitali-Hahn-Saks Theorem together with Dixmier's characterization [3, p. 117] of abelian W^* -algebras gives that $\{\|f_n\|\}$ is bounded on each maximal abelian $*$ subalgebra of M . Since each self-adjoint element of M lies in such a subalgebra, we get (').

Since $\{\|f_n\|\}$ is bounded and the limit of $\{f_n(p)\}$ exists for each p in P , it follows

from the spectral theorem that the limit of $\{f_n(a)\}$ exists for each a in M . Since bounded sets of M^* are weak* relatively compact, there exists f_0 in M^* such that $\{f_n\}$ converges weak* to f_0 . But if $\{f_n\}$ is contained in F , then $\{f_n\}$ is weakly Cauchy, so by Corollary III.3, we have that f_0 lies in F , and $\{f_n\}$ converges weakly to f_0 .

By (") the sequence $\{f_n\}$ is wrc in F . By the proof of Lemma II.3c if $f = \sum_{n=1}^{\infty} 2^{-n} [f_n]$, then given $\alpha > 0$ there is $\beta > 0$ such that if p is in P and $f(p) < \beta$, then $|f_n(p)| < \alpha$, for each $n = 1, 2, \dots$. But if p is in P and $g(p) = 0$, then $f(p) = 0$, so by Proposition 5, p. 62 of [3], we have that given $\beta > 0$ there is $\pi > 0$ such that if p is in P and $g(p) < \pi$, then $f(p) < \beta$. Thus if p is in P and $g(p) < \pi$, then $f(p) < \beta$, so $|f_n(p)| < \alpha$ for $n = 1, 2, \dots$ Q.E.D.

We shall now apply Phillips' Lemma [9] to show that a somewhat weaker version of the Vitali-Hahn-Saks Theorem holds in M^* .

THEOREM III.7. *Let $\{b_n\}$ be a sequence of positive elements of M such that $\sum_{n=1}^{\infty} b_n$ converges in the s -topology. Let $\{f_n\}$ be a sequence in M^* such that $\lim_{n \rightarrow \infty} f_n(p)$ exists for each p in P . Then $\lim_{m \rightarrow \infty} \sum_{n=1}^m |f_k(b_n)|$ exists uniformly for $k = 1, 2, \dots$*

Proof. By [5, p. 199] we get that $\{f_n\}$ is a bounded sequence. Thus by the spectral theorem $\{f_n\}$ is weak* Cauchy, so there exists f in M^* such that $f_n \rightarrow f$ (weak*), by the weak* compactness of bounded sets in M^* . Define measures m_n on the subsets of the positive integers by

$$m_n(J) = (f_n - f) \left(\sum_{k \in J} b_k \right),$$

for a set J of positive integers. Then $\{m_n\}$ are uniformly bounded finitely additive measures and $m_n(J) \rightarrow 0$ for all sets J of positive integers, so we may apply Phillips' Lemma to get $\sum_{k=1}^{\infty} |(f_n - f)(b_k)| \rightarrow 0$ as $k \rightarrow \infty$. But since $\sum_{k=1}^{\infty} b_k$ exists, it follows that $\sum_{k=1}^{\infty} |f(b_k)|$ exists, and hence $\sum_{k=1}^{\infty} |f_n(b_k)|$ exists uniformly for $n = 1, 2, \dots$ Q.E.D.

COROLLARY III.8. *Let $\{f_n\}$ be a sequence in M^* such that $\lim_{n \rightarrow \infty} f_n(p)$ exists for each p in P . If $\{p_k\}$ is an increasing sequence in P , then $\lim_{k \rightarrow \infty} f_n(p_k)$ exists uniformly for $n = 1, 2, \dots$*

Proof. Set $q_k = p_{k+1} - p_k$. Then $\{q_k\}$ is an orthogonal sequence in P , so $\sum_{k=1}^{\infty} q_k$ exists. Apply the theorem to $\{q_k\}$. Q.E.D.

Next we give some characterizations of the weakly relatively compact sets in M^* . Naturally it would be desirable to be able to extend all of the conditions of §II for weak relative compactness in F directly to M^* , with only minor notational changes. This seems to be an extremely difficult problem. It can be shown that suitable modifications of each of the conditions of §II (except (11)) is a necessary condition for weak relative compactness in M^* , but we are able to show sufficiency for only a few of them (notably (3), (4) and (9)). The problem is closely related to

showing that weak* sequential convergence in M^* implies weak sequential convergence in M^* . This is proved in the abelian case by Grothendieck [6].

THEOREM III.9. *A bounded subset K of M^* is wrc iff it satisfies any one of the following conditions:*

- (1) *Both $\{f^n : f \text{ is in } K\}$ and $\{f^s : f \text{ is in } K\}$ are wrc.*
- (2) *There exists positive g in M^* such that given $\alpha > 0$ there is $\beta > 0$ such that if a is in S and $g(aa^* + a^*a) < \beta$, then $|f(a)| < \alpha$ for all f in K .*
- (3) *Same as (2) with a restricted to be self-adjoint.*
- (4) *Same as (2) with a restricted to be positive.*
- (5) *Same as (2) with a restricted to lie in P .*
- (6) *For any sequence $\{f_n\}$ in K there is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that given $\alpha > 0$ there is an integer i_0 and an open set V for the M^* -topology of S (which is non-void) such that if a is in V and $i > i_0, j > i_0$, then $|(f_{n_i} - f_{n_j})(a)| \leq \alpha$.*

Proof. (1) is a simple consequence of [16] and the Eberlein-Smulian Theorem.

By the Takeda Theorem [15], and Theorem II.3, it is clear that (2)–(6) are all necessary. The sufficiency of (2), (3), and (6) is immediate from the Kaplansky Density Theorem, which implies that S (considered as lying in M^{**}) is dense in the $\sigma(M^{**}, M^*)$ -topology of the unit sphere of M^{**} ; so we need only apply [15] and the relevant theorem from §II. Theorem III.2 and the spectral theorem provide the appropriate density conditions for (4) and (5). Q.E.D.

REMARK. It should be noted that conditions (2), (3), (4), and (6) make sense if M is an arbitrary C^* -algebra, and in fact the proof given is clearly valid for this situation. This theorem can then be considered an exact noncommutative generalization of the similar results of [1] for the abelian case.

REMARK. By considering the embedding of M in M^{**} and applying [15], it is easy to see that conditions (5)–(9) of Corollary II.5 are necessary conditions for weak relative compactness in M^* . That they are also sufficient remains a conjecture.

We conclude the section by extending a result of Dixmier [3, p. 56].

COROLLARY III.10. *Let M and N be W^* -algebras and let T be a weak* continuous linear mapping from M to N (e.g., T is a positive normal map). Then T is s^* -continuous on S .*

Proof. It is immediate that T is continuous for the Mackey topologies of M and N . Thus Theorem II.7 applies. Q.E.D.

IV. Some special algebras. In this section we shall restrict attention to some special C^* -algebras and obtain sharper results than in the general case.

THEOREM IV.1. *Suppose M satisfies the condition that for every p in P there exists q in P such that $q \leq p$, $q \neq 0$, and the algebra qMq is finite dimensional. Then given a sequence $\{f_n\}$ of positive functionals in F such that $\lim_{n \rightarrow \infty} f_n(p)$ exists for each p in P , then there exists f in F such that $\lim_{k \rightarrow \infty} \|f_k - f\| = 0$.*

Proof. By Theorem III.6, we know that there exists f in F such that $\{f_k\}$ converges weakly to f in M^* . We need only to show that this implies that $\lim_{k \rightarrow \infty} \|f_k - f\| = 0$. Let $\{p_\theta\}$ be a family of elements of P satisfying

(') The $\{p_\theta\}$ are orthogonal.

(") $\text{lub } \{p_\theta\} = I$.

(") The algebra $p_\theta M p_\theta$ is finite dimensional for each θ . Such a family $\{p_\theta\}$ exists by the hypothesis and Zorn's Lemma. Since the set $\{f_k\}$ together with f is weakly compact, we may apply Theorem II.6 as follows. Given $\alpha > 0$ there is a finite set $p_{\theta_1}, \dots, p_{\theta_r}$ in $\{p_\theta\}$ such that if $p = \text{lub } \{p_{\theta_1}, \dots, p_{\theta_r}\}$, and $p' = I - p$, then $\|R_p L_{p'} f_k\| < \alpha$ for $k = 1, 2, \dots$, and also $\|R_p L_{p'} f\| < \alpha$. Let $\Delta = \sup \{\|f_k\| : k = 1, 2, \dots\}$; then Δ is finite since $\{f_k\}$ is wr. Since pMp is finite dimensional, the weak and the norm topologies coincide on pMp , so there exists an integer k_0 such that if $k > k_0$, then for any a in S

$$\|(f_k - f)(pap)\| < \alpha.$$

Thus for any a in S and $k > k_0$ we have the inequalities

$$\begin{aligned} |(f_k - f)a| &\leq |(f_k - f)(pap)| + |(f_k - f)(pap')| \\ &\quad + |(f_k - f)(p'ap)| + |(f_k - f)(p'ap')| \\ &\leq \alpha + |(f_k - f)(pap')| \\ &\quad + |(f_k - f)(p'ap)| + 2\alpha. \end{aligned}$$

But by the Schwartz inequality we can write

$$\begin{aligned} |(f_k - f)(pap')| &\leq |f_k(pap')| + |f(pap')| \\ &\leq [\Delta f_k(p')]^{1/2} + [\Delta f(p')]^{1/2} \\ &\leq 2(\Delta^{1/2})(\alpha^{1/2}). \end{aligned}$$

Similarly we have

$$|(f_k - f)(p'ap)| \leq 2(\Delta^{1/2})(\alpha^{1/2}).$$

Combining the above estimates we get

$$|(f_k - f)a| \leq 3\alpha + 4(\Delta^{1/2})(\alpha^{1/2}).$$

Since $\alpha > 0$ was arbitrary and a was any element of S we have that

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0. \quad \text{Q.E.D.}$$

REMARK. This theorem can be considered a noncommutative version of [2, p. 33]. It is clear from Example II.10 that we cannot prove a stronger theorem for the general case, at least in the sense that the condition of positivity is essential for the truth of the present theorem.

The next result concerns C^* -algebras which are W^* -algebras only in the finite-dimensional case. We shall consider dual rings (in the sense of Kaplansky [7]) which are C^* -algebras. First we shall need a definition and a result from [7].

DEFINITION. Let $\{H_x\}_{x \in X}$ be a family of Hilbert spaces. Let $C(H_x)$ denote the algebra of compact operators on H_x . The C_∞ direct product of the algebras $\{C(H_x)\}$ is the set A of all elements $\{a_x\}$ in the Cartesian product of the $\{C(H_x)\}$ with the following property: Given $\delta > 0$, then $\{x : \|a_x\| > \delta\}$ is a finite set. A has a natural structure as a C^* -algebra.

THEOREM (KAPLANSKY). *Let A be a dual ring which is also a C^* -algebra. Then there exists a family $\{H_x\}$ of Hilbert spaces such that A is isomorphic to the C_∞ direct product of the $\{C(H_x)\}$ as in the above definition.*

We show that such C^* -algebras share some very strong properties with abelian C^* -algebras. The work for the abelian case was done by Grothendieck [6].

THEOREM IV.2. *Let N be a dual ring which is also a C^* -algebra. Then a bounded set K in N^* is wrc iff for each sequence $\{p_n\}$ of orthogonal self-adjoint projections in N we have that the limit of $\{f(p_n)\} = 0$ uniformly for f in K .*

Proof. Let N be the C_∞ sum of $\{C(H_\theta)\}$. Then it is easy to verify that N^* is isometrically isomorphic to the L_1 direct sum of $\{TC(H_\theta)\}$, where $TC(H_\theta)$ is the Banach space of trace class operators on H_θ . Also the second dual N^{**} of N is seen to be isometrically isomorphic to the complete bounded direct product of $\{B(H_\theta)\}$, where $B(H_\theta)$ is the algebra of all bounded operators on H_θ . Let K be bounded in N^* .

Since N^{**} is a W^* -algebra, if K is assumed to be wrc, the theorem follows from Theorem II.2.

Now assume that K satisfies the condition of the theorem. If K is not wrc, we may apply Theorem II.2 and get that there exists a sequence $\{q_n\}$ of orthogonal self-adjoint projections in N^{**} and $\alpha > 0$ and $\{f_n\}$ in K such that $|f_n(q_n)| > \alpha$ for each $n = 1, 2, \dots$. By the above characterization of N^{**} , for each self-adjoint projection q in N^{**} there exists an orthogonal family $\{p_\varepsilon\}$ of self-adjoint projections in N such that if N is considered as lying in N^{**} , then $\text{lub } \{p_\varepsilon\} = q$. Thus for each $n = 1, 2, \dots$ we may choose a self-adjoint projection p_n in N such that $p_n \leq q_n$ and $|f_n(q_n - p_n)| < \alpha/2$.

Since the $\{q_n\}$ are orthogonal and $p_n \leq q_n$ for each n , we have that the $\{p_n\}$ are also orthogonal. By the above inequalities we may write for each $n = 1, 2, \dots$

$$\alpha/2 > |f_n(q_n - p_n)| \geq ||f_n(q_n)| - |f_n(p_n)||.$$

But by hypothesis there exists n_0 such that if $n > n_0$, then

$$|f_n(p_n)| < \alpha/2.$$

Since $|f_n(q_n)| > \alpha$ for each $n=1, 2, \dots$ we have for $n > n_0$

$$\alpha/2 > |\alpha - \alpha/2| = \alpha/2,$$

a contradiction. Thus K is wr.c. Q.E.D.

THEOREM IV.3. *Let N be a C^* -algebra which is a dual ring. Suppose that T is a bounded linear map of N into a weakly sequentially complete Banach space. Then T is weakly compact.*

Proof. We use the notation of the proof of the last theorem for the structure of N^* and N^{**} . Let T map N into X as in the hypothesis. Consider T^* mapping X^* into N^* . By [4, p. 485] we need only show that T^* is weakly compact. Let K be the unit ball of X^* and let $K' = T^*(K)$, a subset of N^* . By Theorem IV.2, we need only show that if p_n is a sequence of orthogonal self-adjoint projections in N , then $\{f(p_n)\}$ converges to 0 uniformly for f in K' . Let $\{p_n\}$ be such a sequence, and let N' be the C^* -subalgebra of N generated by $\{p_n\}$. Clearly N' is an abelian C^* -algebra. Then by [4, p. 494] we have that the restriction of T to the subalgebra N' is weakly compact. Thus K' restricted to N' is weakly relatively compact. By Theorem II.2 we see that this implies that $\{f(p_n)\}$ converges to 0 uniformly for f in K' . Q.E.D.

In closing we remark that many of the results have applications outside the scope of the present paper. As an example we note that Theorem III.2, combined with the method of [19] yields the following theorem. Let N be an AW^* -algebra and P the set of self-adjoint projections in N . Then N is a W^* -algebra iff there exists a unitarily invariant, separating family K of positive functionals in N^* such that P is complete in the uniformity generated by the seminorms $|f(a^*a)|^{1/2}$ for all f in K .

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